Domain of a function of two variables

Given the following function:

$$f(x,y) = \frac{\sqrt{xy}}{\ln(16 - x^2 - 4y^2)}$$

Determine the domain analytically.

Solution

To determine the domain of the function $f(x, y)$, we need to find all pairs (x, y) for which the expression is defined. We analyze the restrictions imposed by the numerator and the denominator.

- **Restrictions from the numerator $\sqrt{xy}$:**
  - The expression inside the square root must be non-negative:

    $$xy \geq 0$$

    This implies that x and y must both be non-negative or both non-positive.
- **Restrictions from the denominator $\ln(16 - x^2 - 4y^2)$:**
  - The argument of the natural logarithm must be positive:

    $$16 - x^2 - 4y^2 > 0$$

  - Additionally, we must ensure the denominator is not zero:

    $$\ln(16 - x^2 - 4y^2) \neq 0 \implies 16 - x^2 - 4y^2 \neq e^0 = 1$$

Analyzing the restrictions:

1. **Condition 1:** $xy \geq 0$
   - This means x and y have the same sign, or one of them is zero.
2. **Condition 2:** $16 - x^2 - 4y^2 > 0$
   - Rearrange to find the permissible region:

     $$x^2 + 4y^2 < 16$$

     This represents the interior of an ellipse centered at the origin with a semi-major axis $a = 4$ along the x-axis and a semi-minor axis $b = 2$ along the y-axis.
3. **Condition 3:** $16 - x^2 - 4y^2 \neq 1$
   - This excludes points where:

     $$x^2 + 4y^2 = 15$$

     This corresponds to an ellipse slightly smaller than the boundary ellipse of the domain.

Conclusion: The domain D of $f(x, y)$ is:

$$D = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; xy \geq 0,\; x^2 + 4y^2 < 16,\; x^2 + 4y^2 \neq 15 \right\}$$